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Fusion Lemma and Boundary Structure

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The purpose of this paper is to characterize those pairs of compact plane sets which have the fusion property in the sense of the well-known Fusion Lemma of Alice Roth.

1. INTRODUCTION

Let a pair of compact subsets K_1, K_2 of the complex plane $\mathbb{C}$ be given. We say that this pair of sets has the *fusion property* if and only if there exists some positive number $\alpha = \alpha(K_1, K_2)$ such that the following is true: For each pair r_1, r_2 of rational functions and each compact set K there is some rational function r with $|r(z) - r_j(z)| \leqslant \alpha \sup\{ |r_1(w) - r_2(w)| \mid w \in K \cup (K_1 \cap K_2)\}$ for all $z \in K_j \cup K$, simultaneously for $j = 1, 2$.

The following result due to Alice Roth [4] plays a fundamental role in complex approximation.

FUSION LEMMA (cf. [1, p. 113ff]). *Every pair of disjoint compact sets in $\mathbb{C}$ has the fusion property.*

In [1, p. 116], D. Gaier asked whether the assumption $K_1 \cap K_2 = \varnothing$ of the Fusion Lemma could be replaced by a weaker condition. An example of P. Gauthier shows that it is no longer true when this assumption is simply omitted. While Gauthier's example deals with rather complicated sets, Gaier himself later published an example for a pair of compact sets K_1, K_2 without the fusion property where K_1, K_2 are simply squares with a common edge [2]. The purpose of this paper is to characterize those pairs of compact plane sets which have the fusion property, provided that some additional topological conditions are fulfilled. For a set $M \subset \mathbb{C}$ let

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M^c, $\mathring{M}$, $\bar{M}$, and ∂M denote the complement, the interior, the closure, and the boundary, respectively. We call a pair of sets $K_1, K_2 \subset \mathbb{C}$ *normal* iff

(n1) K_1 and K_2 are compact,

(n2) $(K_1 \cup K_2)^c$ is connected,

(n3) $\mathring{K}_1$ and $\mathring{K}_2$ are connected,

(n4) $K_1 \cap \mathring{K}_2 = K_2 \cap \mathring{K}_1 = \varnothing$,

(n5) $K_1 = \overline{\mathring{K}_1}$ and $K_2 = \overline{\mathring{K}_2}$.

We note that the latter two conditions are only technical and could be omitted by using additional but standard arguments in the proof of our main result. Also we remark that (n3) can be replaced by

(n3′) $\mathring{K}_1$ and $\mathring{K}_2$ are simply connected domains

as follows from (n3) by use of (n2), (n4), and (n5).

Our main result gives a purely topological characterization of those normal pairs which have the fusion property:

THEOREM. *For any normal pair $K_1, K_2 \subset \mathbb{C}$ the following conditions are equivalent*:

(i) *K_1, K_2 have the fusion property,*

(ii) $\partial K_1 \cup \partial K_2 = \partial(K_1 \cup K_2)$.

This result comes out by a combination of the idea of Gaier used in [2] together with an observation of Schmieder and Shiba [5].

Finally we discuss the assumption of normality we have made in our theorem. It is not at all obvious that (ii) no longer implies (i) if (n2) is omitted from our list above. The first version of this paper was written in Oberwolfach on February 14, 1990. Some weeks later I became acquainted with an article of A. A. Nersesyan [3], where similar problems are treated. Nersesyan's results are in some sense more general, but on the other hand more special (he only considers the case $K = K_1 \cap K_2$). Furthermore, the methods are rather different (cf. [5]).

2. Fusion Property Implies Boundary Property

In this section we shall prove the part (i) $\Rightarrow$ (ii) or our theorem. We start with some preliminary considerations. Let a pair K_1 and K_2 of normal compact planar sets be fixed.

Suppose there is some bounded component of $(\partial K_1 \cap \partial K_2)^c$. From (n2) we see that this component must be contained in $K_1 \cup K_2$, and from (n4) we obtain that it must be covered either by K_1 or by K_2. Therefore we conclude from (n3) that $(\partial K_1 \cap \partial K_2)^c$ can have at most two bounded components (in addition to the unbounded one). By a well-known result of Mergelyan (cf. [1, p. 110]) every continuous function $\partial K_1 \cap \partial K_2 \to \mathbb{C}$ can be uniformly approximated on $\partial K_1 \cap \partial K_2$ by rational functions. From the Tietze Extension Theorem it follows that every compact subset of $\partial K_1 \cap \partial K_2$ has this approximation property as well.

If (ii) in the theorem were false, we would have

$$\partial K_1 \cup \partial K_2 \supsetneqq \partial(K_1 \cup K_2).$$

So we can choose some $\zeta_0 \in \partial K_1 \cup \partial K_2$ which is an interior point of $K_1 \cup K_2$. By (n4), ζ_0 can be neither an interior point of K_1 nor of K_2 which implies that $\zeta_0 \in \partial K_1 \cap \partial K_2$. Since $\zeta_0 \in \overset{\circ}{\overline{K_1 \cup K_2}}$, there is an $0 < \varepsilon < 2$ such that $U_\varepsilon(\zeta_0) \cap (\partial K_1 \cap \partial K_2) \subset \overset{\circ}{\overline{K_1 \cup K_2}}$. The set $A := \overline{U_\varepsilon(\zeta_0)} \cap (\partial K_1 \cap \partial K_2)$ has the above mentioned approximation property.

Now fix two points $z_1, z_2 \in A - \{\zeta_0\}$. Then for every $\delta > 0$ there are suitable points (using (n5))

$$w_1^1 \in U_\delta(z_1) \cap \overset{\circ}{K}_1, \qquad w_2^1 \in U_\delta(z_1) \cap \overset{\circ}{K}_2,$$

$$w_1^2 \in U_\delta(z_2) \cap \overset{\circ}{K}_1, \qquad w_2^2 \in U_\delta(z_2) \cap \overset{\circ}{K}_2.$$

By (n3) there is some Jordan arc γ_1 in $\overset{\circ}{K}_1$ joining w_1^1, w_1^2 as well as some Jordan arc γ_2 in $\overset{\circ}{K}_2$ joining w_2^1, w_2^2. Without loss of generality we may assume that $C := (\overline{U_\delta(z_1)} \cup \overline{U_\delta(z_2)} \cup \gamma_1 \cup \gamma_2)^c$ is the union of exactly two components. This follows by a slight modification of the Jordan Curve Theorem. This is true for all δ sufficiently small for suitably chosen curves γ_1, γ_2. The bounded component of C must contain a connected part $B \subset A$ whose boundary meets $U_\delta(z_1)$ as well as $U_\delta(z_2)$.

Now we take some $z_0 \in B$ and claim that for every $\delta > 0$ there is a smooth Jordan arc $\Gamma_\delta\colon [-1, 1] \to B_\delta := \{z \mid \operatorname{dist}(z, B) > \delta\}$ with $\Gamma_\delta(-1) = z_1$, $\Gamma_\delta(0) = z_0$, $\Gamma_\delta(1) = z_2$.

Now the Lemma used by Gaier in [2] appears in the following form:

LEMMA. *For every $M > 0$ there exists some polynomial P with*

(1) $P(z_0) = 0$,

(2) $|P(z)| \leqslant 1$ *for all* $z \in A$,

(3) *There is some $\delta_0 > 0$ such that, for every $0 < \delta < \delta_0$ and every smooth Jordan arc $\Gamma_\delta\colon [-1, 1] \to B_\delta$ which joins z_1, z_0, z_2 as above, the estimate*

$$\left| \int_{\Gamma_\delta} \frac{P(z)}{z - z_0}\, dz \right| \geqslant M$$

is true.

Proof. We extend the idea of Gaier [2].

Let $\eta > 0$ be arbitrary for the present and define $h_\eta(z) := 1/|z - z_0|$ for all $z \in A$ with $|z - z_0| \geqslant \eta$ and $h_\eta(z) := 1/\eta$ for all $z \in A \cap U_\eta(z_0)$.

As mentioned above, A has the approximation property. So we can find a polynomial $p(z, \eta)$ in z with $|p(z, \eta) - h_\eta(z)| < 1/2$ for all $z \in A$.

Now for every $\eta > 0$ we choose some $\delta_0 = \delta_0(\eta)$ such that for every pair $z \in \mathbb{C}$, $w \in A$ with $|z - w| < \delta_0$ we have the estimate $|p(z, \eta) - p(w, \eta)| < 1/2$.

For fixed η we now consider some positive number $\delta < \delta_0(\eta)$ and some arc Γ_δ as described above. We denote by L_α the length of that part of Γ_δ with $|\Gamma_\delta - z_0| \leqslant \alpha < \eta$. Then we obtain

$$\begin{aligned}
&\left| \int_{\Gamma_\delta} p(z, \eta)\, dz \right| \\
&\quad = \left| \int_{|\Gamma_\delta - z_0| < \alpha} p(z, \eta)\, dz + \int_{|\Gamma_\delta - z_0| \geqslant \alpha} p(z, \eta)\, dz \right| \\
&\quad = \left| \int_{|\Gamma_\delta - z_0| \leqslant \alpha} p(z, \eta) - p(z_0, \eta) + p(z_0, \eta)\, dz + \int_{|\Gamma_\delta - z_0| \geqslant \alpha} p(z, \eta)\, dz \right| \\
&\quad \geqslant \left| \int_{|\Gamma_\delta - z_0| \geqslant \alpha} p(z, \eta)\, dz \right| - \left| \int_{|\Gamma_\delta - z_0| \leqslant \alpha} p(z_0, \eta)\, dz \right| \\
&\qquad - \left| \int_{|\Gamma_\delta - z_0| \leqslant \alpha} p(z, \eta) - p(z_0, \eta)\, dz \right| \\
&\quad = \left| \int_{|\Gamma_\delta - z_0| \geqslant \alpha} p(z, \eta)\, dz \right| - \left| \int_{|\Gamma_\delta - z_0| \leqslant \alpha} p(z_0, \eta) - h_\eta(z_0) + \frac{1}{\eta}\, dz \right| \\
&\qquad - \left| \int_{|\Gamma_\delta - z_0| \leqslant \alpha} p(z, \eta) - p(z_0, \eta)\, dz \right| \\
&\quad \geqslant \left| \int_{|\Gamma_\delta - z_0| \geqslant \alpha} h_\eta(z)\, dz \right| - \left| \int_{|\Gamma_\delta - z_0| \geqslant \alpha} \frac{1}{2}\, dz \right| - \left(\frac{1}{2} + \frac{1}{\eta} + \frac{1}{2} \right) L_\alpha.
\end{aligned}$$

The sum in the last line tends to infinity when η tends to 0 and $\alpha < \eta$ has been chosen small enough. This follows from the definition of h_η by an elementary consideration—note that for small α we have $L_\alpha \approx 2\alpha < 2\eta$. This works for all $\delta < \delta_0(\eta)$.

Therefore we can fix some η such that for every $\delta < \delta_0(\eta)$ we have

$$\left| \int_{\Gamma_\delta} p(z, \eta)\, dz \right| > 2M. \qquad (*)$$

For this η we now define $P(z) := ((z - z_0)/2)\, p(z, \eta)$. So (1) is fulfilled.

The following inequalities are valid for all $z \in A$:

$$|P(z)| \leqslant \frac{1}{2}|z-z_0|\,(|p(z,\eta)-h_\eta(z)|+|h_\eta(z)|)$$

$$<\frac{1}{2}|z-z_0|\left(\frac{1}{2}+\frac{1}{|z-z_0|}\right)$$

$$<\frac{\varepsilon}{4}+\frac{1}{2}<1$$

which gives (2). Finally (3) is an immediate consequence of (*), and the proof of our lemma is complete.

Now we can give the proof for the direction (i) ⇒ (ii) in our theorem. We assume that some normal pair K_1, K_2 is given such that (ii) is violated. As described above, we fix points $z_0, z_1, z_2 \in A$ and Jordan arcs γ_1, γ_2 in $\mathring{K}_1$ resp. $\mathring{K}_2$ as mentioned. Furthermore we choose according to our lemma some smooth Jordan arc Γ_δ which joins z_1 and z_2 via z_0. Now we extend γ_1 and γ_2 by adding straight lines which join z_1 with w_1^1 resp. w_2^1 as well as z_2 with w_1^2 resp. w_2^2. Let the resulting arcs be denoted by Γ_1 and Γ_2. Call their lengths L_1 and L_2, resp., and define

$$d_j = \frac{1}{\operatorname{dist}(\Gamma_j, z_0)} \qquad (j=1, 2).$$

We now assume that the pair K_1, K_2 has the fusion property in contradiction to our theorem. Let the related number be $\alpha = \alpha(K_1, K_2)$. For $M := 8\alpha(L_1 d_1 + L_2 d_2)$ we fix some polynomial P according to the lemma.

The arcs Γ_1, Γ_2 can be joined to a closed Jordan curve whose complement has the bounded component Ω. We consider the set $\Omega_1 = \overline{K_1 \cap \Omega} \cup \partial K_1$. From the concept of normality it follows that K_1^c consists of at most two components. By use of (n3) and (n5) we conclude that $(\partial K_1)^c$ has at most three components.

Now from the definition of Ω_1 we see that if Ω_1^c has infinitely many components then all but finitely many of them must meet $U_\delta(z_1)$ or $U_\delta(z_2)$. So, if some components of Ω_1^c have a diameter $\leqslant \delta$, then almost all of them are contained in $\overline{U_{2\delta}(z_1)} \cup \overline{U_{2\delta}(z_2)} =: V_\delta$.

Without loss of generality we may choose δ so small that $|P(z)| < 3/2$ holds for all $z \in V_\delta$. Now, if we take N_1 to be the union of Ω_1 with the closure of all the components of Ω_1^c which have diameter $\leqslant \delta$, then N_1 has the approximation property by a result of Mergelyan (cf. [1, p. 110]): Every continuous function $f\colon N_1 \to \mathbb{C}$ which is holomorphic in the interior of N_1 can be approximated uniformly on N_1 by rational functions.

We can construct such a function f as follows: Let $f \mid \overset{\circ}{\bar{N}}_1 = P$ and extend f continuously to N_1 by the Tietze Extension Theorem so that $|f(z)| \leqslant 3/2$ for all $z \in \partial K_1 \cap \partial K_2$.

Then we can find a rational function r_1 with

$$|r_1(z)| \leqslant 2 \qquad \text{for all} \quad z \in \partial K_1 \cap \partial K_2 = K_1 \cap K_2. \tag{$**$}$$

With the data $r_2 \equiv 0$, $K = K_1 \cap K_2$ we now apply the fusion property to get a rational function r with

$$|r(z) - P(z)| \leqslant 2\alpha \qquad \text{for all} \quad z \in K_1,$$
$$|r(z) - 0| \leqslant 2\alpha \qquad \text{for all} \quad z \in K_2.$$

For the rest of the proof we can follow Gaier [2] again.

Let $R(z) := r(z) - r(z_0)$. Then we have

$$0 = \int_{\Gamma_1 \cup \Gamma_2} \frac{R(z)}{z - z_0}\, dz = \int_{\Gamma_1} \frac{R(z) - P(z)}{z - z_0}\, dz + \int_{\Gamma_1} \frac{P(z)}{z - z_0}\, dz + \int_{\Gamma_2} \frac{P(z)}{z - z_0}\, dz$$

as well as $|R - P| \leqslant 4\alpha$ on K_1 and $|R| \leqslant 4\alpha$ on K_2. Thus we obtain

$$\left| \int_{\Gamma_1} \frac{R(z) - P(z)}{z - z_0}\, dz \right| \leqslant 4\alpha L_1 d_1 \qquad \text{and} \qquad \left| \int_{\Gamma_2} \frac{R(z)}{z - z_0}\, dz \right| \leqslant 4\alpha L_2 d_2.$$

Further it follows from $P(z_0) = 0$ that

$$\int_{\Gamma_1} \frac{P(z)}{z - z_0}\, dz = \int_{\Gamma_1 \cup \Gamma_\delta} \frac{P(z)}{z - z_0}\, dz - \int_{\Gamma_\delta} \frac{P(z)}{z - z_0}\, dz = -\int_{\Gamma_\delta} \frac{P(z)}{z - z_0}\, dz.$$

This leads to

$$\left| \int_{\Gamma_\delta} \frac{P(z)}{z - z_0}\, dz \right| \leqslant \left| \int_{\Gamma_1} \frac{R(z) - P(z)}{z - z_0}\, dz \right| + \left| \int_{\Gamma_2} \frac{P(z)}{z - z_0}\, dz \right|$$
$$\leqslant 4\alpha(L_1 d_1 + L_2 d_2) < M$$

which contradicts the estimate (3) of our lemma.

Boundary Property Implies Fusion Property

The remaining direction (ii) ⇒ (i) of our theorem is an immediate consequence of a result of Schmieder and Shiba [5, Sect. 4].

Finally we discuss the assumption (n2) that $(K_1 \cup K_2)^c$ is connected. By a rather obvious cutting argument it would not be hard to prove the theorem under the weaker condition that $(K_1 \cup K_2)^c$ has finitely many components (while retaining the other assumptions). But this conclusion becomes false in the case when $(K_1 \cup K_2)^c$ has infinitely many components. This is shown by the example of Gauthier mentioned above (cf. [1, p. 116ff]), where $K_1 \cup K_2$ is the so-called "stitched disc."

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